Exact asymptotics for linear processes

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October 2011

Plan of talk

-Early results

-Central limit theorem for linear processes

-Functional central limit theorem for linear processes

-Selfnormalized CLT

-Exact asymptotic for linear processes

Early results: i.i.d. finite second moment

Theorem

Let (ξ_j) be i.i.d., centered at expectation with finite second moments.

$$\frac{\sum_{j=1}^{n} \xi_j}{\sqrt{n}} \longrightarrow \sigma N(0,1)$$

and

$$\frac{\sum_{j=1}^{[nt]} \xi_j}{\sqrt{n}} \longrightarrow \sigma W(t)$$

Here $\sigma = stdev(\xi_0)$.

CLT for linear processes with finite second moments

$$X_k = \sum_{j=-\infty}^{\infty} a_{k+j}\xi_j, \; S_n = \sum_{j=1}^{n} X_j,$$

Theorem

(Ibragimov and Linnik, 1971) Let (ξ_j) be i.i.d. centered with finite second moment, $\sum_{k=-\infty}^{\infty} a_k^2 < \infty$ and $\sigma_n^2 = var(S_n) \to \infty$. Then

$$S_n/\sigma_n \xrightarrow{D} N(0,1).$$

$$\sigma_n^2 = \sum_{j=-\infty}^{\infty} b_{nj}^2 \;, \; b_{n,j} = a_{j+1} + \ldots + a_{j+n}.$$

It was conjectured that a similar result might hold without the assumption of finite second moment.

Functional central limit theorem question for linear processes.

For $0 \leq t \leq 1$ define

$$W_n(t) = \frac{\sum_{i=1}^{[nt]} X_i}{\sigma_n}$$

where $[x]$ is the integer part of x.

Problem

Let (ξ_j) be i.i.d. centered with finite second moment, $\sum_{k=-\infty}^{\infty} a_k^2 < \infty$ and $\sigma_n^2 = nh(n)$ with $h(x)$ a function slowly varying at ∞.$(h(tx)/h(x) \to 1$ for all t as $x \to \infty)$. Is it true that $W_n(t) \Rightarrow W(t)$, where $W(t)$ is the standard Brownian motion?

This will necessarily imply in particular that for every $\varepsilon \geq 0$,

$$\mathbb{P}(\max_{1 \leq i \leq n} |X_i| \geq \varepsilon\sigma_n) \to 0 \text{ as } n \to \infty.$$

Functional CLT. Counterexample.

Example

There is a linear process (X_k) such that $\sigma_n^2 = nh(n)$ and such that the weak invariance principle does not hold:

$$\mathbb{P}(|\xi_0| > x) \sim \frac{1}{x^2 \log^{3/2} x},$$

$$a_0 = 0, a_1 = \frac{1}{\log 2} \text{ and } a_n = \frac{1}{\log(n+1)} - \frac{1}{\log n}, \text{ for } n \geq 2,$$

$$\sigma_n^2 \sim n/(\log n)^2 \text{ and } \limsup_{n\to\infty} \mathbb{P}\big(\max_{1\leq i\leq n} |\xi_i| \geq \varepsilon\sigma_n\big) = 1\,.$$

Functional CLT.

When $\mathbb{E}(|\xi_0|^{2+\delta}) < \infty$ and $\sigma_n^2 = nh(n)$ then functional CLT holds.
$W_n(t) \to W(t)$ with $W_n(t)$ standard Brownian motion
Merlevède-P (2006).

When $\mathbb{E}(\xi_0^2) < \infty$ and $\sigma_n^2 = n^\lambda h(n)$ with $\lambda > 1$ then $W_n(t)$ converges weakly to the fractional Brownian motion W_H with Hurst index $\lambda/2$.

Fractional Brownian motion with Hurst index $\lambda/2$, i.e. is a Gaussian process with covariance
structure $\frac{1}{2}(t^\lambda + s^\lambda - (t-s)^\lambda)$ for $0 \leq s < t \leq 1$.

CLT for i.i.d. centered with infinite second moments

$$(*) \qquad H(x) = \mathbb{E}(\xi_0^2 I(|\xi_0| \le x)) \text{ is a slowly varying function at } \infty.$$

Define $b = \inf\{x \ge 1 : H(x) > 0\}$

$$\eta_j = \inf\left\{s : s \ge b+1,\ H(s)/s^2 \le j^{-1}\right\}, \quad j = 1, 2, \cdots$$

Theorem

Then

$$\frac{\sum_{j=1}^{n} \xi_j}{\sqrt{nH_n}} \to N(0,1)$$

and

$$\frac{\sum_{j=1}^{[nt]} \xi_j}{\sqrt{nH_n}} \to W(t)$$

where $H_n = H(\eta_j)$

Selfnormalized CLT for i.i.d. centered with infinite second moments

Giné, Götze and Mason(1997)

Theorem

$H(x) = \mathbb{E}(\xi_0^2 I(|\xi_0| \leq x))$ is a slowly varying function at ∞ is equivalent to

$$\frac{\sum_{j=1}^{n} \xi_j}{\sqrt{\sum_{j=1}^{n} \xi_j^2}} \to N(0,1)$$

and

$$\frac{\sum_{j=1}^{[nt]} \xi_j}{\sqrt{\sum_{j=1}^{n} \xi_j^2}} \to W(t)$$

where $H_n = H(\eta_j)$

CLT for linear processes with infinite second moments

$X_0 = \sum_{j=-\infty}^{\infty} a_j \xi_j$ is well defined if

$$\sum_{j \in \mathbb{Z}, a_j \neq 0} a_j^2 H(|a_j|^{-1}) < \infty,$$

Theorem

(P-Sang, 2011) Let $(\xi_k)_{k\in\mathbb{Z}}$ be i.i.d., centered. Then the following statements are equivalent:
(1) ξ_0 is in the domain of attraction of the normal law (i.e. satisfies $(*)$)
(2) For any sequence of constants $(a_n)_{n\in\mathbb{Z}}$ as above and $\sum_{j=-\infty}^{\infty} b_{nj}^2 \to \infty$ the CLT holds. (i.e. there are constants D_n such that $S_n / D_n \to N(0,1)$).

Regular weights and infinite variance (long memory).

$$a_n = n^{-\alpha} L(n), \text{ where } 1/2 < \alpha < 1 \; ,$$
$$\mathbb{E}(\xi_0^2 I(|\xi_0| \le x)) = H(x)$$

Example

Fractionally integrated processes. For $0 < d < 1/2$ define

$$X_k = (1-B)^{-d} \xi_k = \sum_{i \ge 0} a_i \xi_{k-i} \text{ where } a_i = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}$$

and B is the backward shift operator, $B\varepsilon_k = \varepsilon_{k-1}$.

For any real x, $\lim_{n\to\infty} \Gamma(n+x)/n^x\Gamma(n) = 1$ and so

$$\lim_{n\to\infty} a_n/n^{d-1} = 1/\Gamma(d).$$

Regularly varying weights and infinite variance; normalizers.

Define $b = \inf\{x \geq 1 : H(x) > 0\}$

$$\eta_j = \inf\left\{s : s \geq b+1,\ H(s)/s^2 \leq j^{-1}\right\}, \quad j = 1, 2, \cdots$$

$$B_n^2 := c_\alpha H_n n^{3-2\alpha} L^2(n) \text{ with } H_n = H(\eta_n)$$

where

$$c_\alpha = \{\int_0^\infty [x^{1-\alpha} - \max(x-1,0)^{1-\alpha}]^2 dx\}/(1-\alpha)^2 \ .$$

Invariance principle for regular weights and infinite variance (long memory).

$a_n = n^{-\alpha} L(n)$, where $1/2 < \alpha < 1$, $n \geq 1$, $\mathbb{E}(\xi_0^2 I(|\xi_0| \leq x)) = H(x)$, $L(n)$ and $H(x)$ are both slowly varying at ∞.

Theorem

(P-Sang 2011) Define $W_n(t) = S_{[nt]}/B_n$. Then, $W_n(t)$ converges weakly to the fractional Brownian motion W_H with Hurst index $3/2 - \alpha$, $(1/2 < \alpha < 1)$.

Fractional Brownian motion with Hurst index $3/2 - 2\alpha$, i.e. is a Gaussian process with covariance structure $\frac{1}{2}(t^{3-2\alpha} + s^{3-2\alpha} - (t-s)^{3-2\alpha})$ for $0 \leq s < t \leq 1$.

Selfnormalized invariance principle

Theorem

(P-Sang 2011) Under the same conditions we have

$$\frac{1}{nH_n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} A^2 \text{ where } A^2 = \sum_i a_i^2$$

and therefore

$$\frac{S_{[nt]}}{na_n\sqrt{\sum_{i=1}^{n} X_i^2}} \Rightarrow \frac{\sqrt{c_\alpha}}{A} W_H(t) \ .$$

In particular

$$\frac{S_n}{na_n\sqrt{\sum_{i=1}^{n} X_i^2}} \Rightarrow N(0, \frac{c_\alpha}{A^2}) \ .$$

We aim to find a function $N_n(x)$ such that, as $n \to \infty$,

$$\frac{\mathbb{P}(S_n \geq x\sigma_n)}{N_n(x)} = 1 + o(1), \text{ with } \sigma_n^2 = \|S_n\|_2^2.$$

where $x = x_n \geq 1$ (Typically $x_n \to \infty$).

We call $\mathbb{P}(S_n \geq x_n\sigma_n)$ the probability of *moderate* or *large deviation* probabilities depending on the speed of $x_n \to \infty$.

Exact asymptotics versus logarithmic

Exact approximation is more accurate and holds under less restrictive moment conditions than the logarithmic version

$$\frac{\log \mathbb{P}(S_n \geq x\sigma_n)}{\log N_n(x)} = 1 + o(1).$$

For example, suppose $\mathbb{P}(S_n \geq x\sigma_n) = 10^{-4}$ and $N_n(x) = 10^{-5}$; then their logarithmic ratio is 0.8, which does not appear to be very different from 1, while the ratio for the exact version is as big as 10.

Nagaev Result for i.i.d.

Theorem

(Nagaev, 1979) Let (ξ_i) *be i.i.d. with*

$$\mathbb{P}(\xi_0 \geq x) = \frac{h(x)}{x^t}(1 + o(1)) \text{ as } x \to \infty \text{ for some } t > 2,$$

and for some $p > 2$, ξ_0 *has absolute moment of order* p*. Then*

$$\mathbb{P}(\sum_{i=1}^{n} \xi_i \geq x\sigma_n) = (1 - \Phi(x))(1 + o(1)) + n\mathbb{P}(\xi_0 \geq x\sigma_n)(1 + o(1))$$

for $n \to \infty$ *and* $x \geq 1$*.*

Nagaev Result for i.i.d.

Notice that in this case

$$N_n(x) = (1 - \Phi(x)) + n\mathbb{P}(\xi_0 \geq x\sigma_n).$$

If $1 - \Phi(x) = o[n\mathbb{P}(\xi_0 \geq x\sigma_n)]$ then in we can also choose $N_n(x) = 1 - \Phi(x)$.

If $n\mathbb{P}(\xi_0 \geq x\sigma_n) = o(1 - \Phi(x))$ we have $N_n(x) = n\mathbb{P}(\xi_0 \geq x\sigma_n)$. The critical value of x is about $x_c = (2\log n)^{1/2}$.

Let (ξ_i) be i.i.d. with

$$\mathbb{P}(\xi_0 \geq x) = \frac{h(x)}{x^t}(1+o(1)) \text{ as } x \to \infty \text{ for some } t > 2,$$

and for some $p > 2$, ξ_0 has absolute moment of order p.

Theorem

(P-Sang-Zhong-Wu, 2011) Let $S_n = \sum_{i=1}^n X_i$ where X_i is a linear process. Then, as $n \to \infty$,

$$\mathbb{P}\left(S_n \geq x\sigma_n\right) = (1+o(1)) \sum_{i=-\infty}^{\infty} \mathbb{P}(b_{n,i}\xi_0 \geq x\sigma_n) + (1-\Phi(x))(1+o(1))$$

holds for all $x > 0$ when $\sigma_n \to \infty$, $\sum_{k=-\infty}^{\infty} a_k^2 < \infty$ and $b_{nj} > 0$,

$$b_{n,j} = a_{j+1} + \cdots + a_{j+n}.$$

Zones of moderate and large deviations

Define the Lyapunov's proportion

$$D_{nt} = B_{n2}^{-t/2} B_{nt} \text{ where } B_{nt} = \sum_i b_{ni}^t.$$

For $x \geq a(\ln D_{nt}^{-1})^{1/2}$ with $a > 2^{1/2}$ we have

$$\mathbb{P}(S_n \geq x\sigma_n) = (1 + o(1)) \sum_{i=1}^{k_n} \mathbb{P}(c_{ni}\xi_0 \geq x\sigma_n) \text{ as } n \to \infty \ .$$

On the other hand, if $0 < x \leq b(\ln D_{nt}^{-1})^{1/2}$ with $b < 2^{1/2}$, we have

$$\mathbb{P}\left(S_n \geq x\sigma_n\right) = (1 - \Phi(x))(1 + o(1)) \text{ as } n \to \infty.$$

Application

Value at risk (VaR) and expected shortfall (ES) are equivalent to quantiles and tail conditional expectations.
Under the assumption $\lim_{x\to\infty} h(x) \to h_0 > 0$

$$\mathbb{P}\left(S_n \geq x\sigma_n\right) = (1 + o(1))\frac{h_0}{x^t}D_{nt} + (1 - \Phi(x))(1 + o(1)).$$

Given $\alpha \in (0, 1)$, let $q_{\alpha,n}$ be defined by $\mathbb{P}(S_n \geq q_{\alpha,n}) = \alpha$.
$q_{\alpha,n}$ can be approximated by $x_\alpha \sigma_n$ where $x = x_\alpha$ is the solution to the equation

$$\frac{h_0}{x^t}D_{nt} + (1 - \Phi(x)) = \alpha.$$

Extension to dependent structures

-CLT for stationary and ergodic differences innovations with finite second moment. (P-Utev, 2006)
-invariance principles for generalized martingales Wu Woodroofe (2004), Dedecker-Merlevède-P (2011)
-moderate deviations for generalized martingales. Merlevède-P (2010)
- CLT stationary martingales differences with infinite second moment plus a mild mixing assumption. (P-Sang 2011)

Results for mixing sequences under various mixing assumptions.

Some open problems

Is the CLT for linear processes equivalent with its selfnormalized version?

$$S_n / V_n \to N(0,1) \text{ where } V_n^2 = \sum_{i=1}^{n} X_i^2$$

CLT for linear processes with infinite variance and ergodic martingale innovations

Functional CLT for linear processes with i.i.d. innovations finite second moment and $var(S_n) = nh(n)$
(necessary and sufficient conditions on the constants)

The same question for generalized martingales

Exact asymptotics for classes of Markov chains

More classes of functions of linear processes

Referrences

Peligrad, Magda; Sang, Hailin Asymptotic Properties of Self-Normalized Linear Processes (2011); to appear in Econometric Theory. arXiv:1006.1572

Peligrad, Magda; Sang, Hailin Central limit theorem for linear processes with infinite variance. (2011); to appear in J. Theoret. Probab. arXiv:1105.6129

Peligrad, Magda; Sang, Hailin; Zhong, Yunda; Wu, Wei Biao. Exact Moderate and Large Deviations for Linear Processes (2011).

Key Ingredients for exact deviations.

Lemma

Assume $S_n = \sum_{j=1}^{k_n} X_{nj}$ (X_{nj} triangular array of independent variables) is stochastically bounded, the variables are centered, and $x_n \to \infty$. Then for any $0 < \eta < 1$, and $\varepsilon > 0$ such that $1 - \eta > \varepsilon$,

$$\mathbb{P}(S_n \geq x_n) = \mathbb{P}(S_n^{(\varepsilon x_n)} \geq x_n) + \sum_{j=1}^{k_n} \mathbb{P}(X_{nj} \geq (1-\eta)x_n)$$

$$+o(\sum_{j=1}^{k_n} \mathbb{P}(X_{nj} \geq \varepsilon x_n)) + \sum_{j=1}^{k_n} \mathbb{P}((1-\eta)x_n \leq X_{nj} < (1+\eta)x_n).$$

Fuk–Nagaev inequality (S. Nagaev, 1979)

Theorem

Let $Y_1, Y_2, \cdots, Y_n$ *be independent random variables and* $m \geq 2$. *Suppose* $\mathbb{E} Y_i = 0$, $i = 1, \cdots, n$, $\beta = m/(m+2)$, *and* $\alpha = 1 - \beta = 2/(m+2)$. *For* $y > 0$, *define* $Y^{(y)} = Y_i I(Y_i \leq y)$,
$A_n(m; 0, y) := \sum_{i=1}^n \mathbb{E}[Y_i^m I(0 < Y_i < y)]$ *and*
$B_n^2(-\infty, y) := \sum_{i=1}^n \mathbb{E}[Y_i^2 I(Y_i < y)]$. *Then for any* $x > 0$ *and* $y > 0$

$$\mathbb{P}(\sum_{i=1}^n Y_i^{(y)} \geq x) \leq \exp(-\frac{\alpha^2 x^2}{2e^m B^2(-\infty, y)}) + (\frac{A(m; 0, y)}{\beta x y^{m-1}})^{\beta x / y}.$$

Theorem

Let $(X_{nj})_{1\le j\le k_n}$ *be an array of row-wise independent centered random variables. Let* $p>2$ *and denote* $S_n=\sum_{j=1}^{k_n}X_{nj}$, $\sigma_n^2=\sum_{j=1}^{k_n}\mathbb{E}X_{nj}^2\to\infty$, $M_{np}=\sum_{j=1}^{k_n}\mathbb{E}X_{nj}^pI(X_{nj}\ge 0)<\infty$, $L_{np}=\sigma_n^{-p}M_{np}$ *and denote*

$$\Lambda_n(u,s,\epsilon)=\frac{u}{\sigma_n^2}\sum_{j=1}^{k_n}\mathbb{E}X_{nj}^2I(X_{nj}\le -\epsilon\sigma_n/s).$$

Furthermore, assume $L_{np}\to 0$ *and* $\Lambda_n(x^4,x^5,\epsilon)\to 0$ *for any* $\epsilon>0$. *Then if* $x\ge 0$ *and* $x^2-2\ln(L_{nt}^{-1})-(t-1)\ln\ln(L_{nt}^{-1})\to-\infty$, *we have*

$$\mathbb{P}\left(S_n\ge x\sigma_n\right)=(1-\Phi(x))(1+o(1)).$$